On dual descriptions of intermittency in a jet *

R. Peschanski

CEA, Service de Physique Theorique, CE-Saclay
F-91191 Gif-sur-Yvette Cedex, France

February 1, 2008

Abstract

Models of intermittent behaviour are usually formulated using a set of multiplicative random weights on a Cayley tree. However, intermittency in particle multiproduction from QCD jets is related to fragmentation of an additive quantum number, e.g. energy-momentum. We exhibit the non-trivial stochastic mapping between these *additive* and *multiplicative* cascading processes.

1. Introduction: Intermittency and its formal description in terms of mutiplicative cascading

When Andrzej Bialas and I, 10 years ago, were puzzled by the mysterious multiplicity fluctuations observed in an ultra-energetic cosmic ray event (JACEE collaboration), we were far from realizing that a systematic study of multiparticle production processes would emerge from considering the factorial moments of the multiplicities distributions. The proposal we made[1] was guided only by two seemingly reasonable requirements:

i) obtaining a non-subjective measure of fluctuations using the moments of the multiplicity distribution,

ii) eliminating (as much as possible) the obvious source of statistical fluctuations due to the finite number of particles produced by event.

We thus proposed the measurement of normalized factorial moments. Under the (strong!) hypothesis that the statistical noise was simple, i.e.

---

*Written in honour of Andrzej Bialas for his 60th birthday and ever young passion for Physics.

Poissonian in the following formula (it was assumed Binomial in the case of the JACEE event because of the fixed multiplicity), one writes:

$$F_q(m) \equiv \frac{\langle n(n-1)...(n-q+1)\rangle_m}{\langle n\rangle_m^q} = \frac{\langle \rho^q\rangle_m}{\langle \rho\rangle_m^q},\qquad(1)$$

where m labels a (suitably chosen) piece of available phase-space for the reaction products, n is the number of particles registered in the bin for one event and ρ is by definition an associated continuous dynamical variable corresponding to the local multiplicity density.

It is quite clear that the absence of fluctuations other than Poissonian would give $F_q(m) = 1$. Moreover, varying the binsize m allows one to look deeper into short-distance fluctuations without being hidden by the set of increasingly erratic statistical fluctuations. In the same spirit, one would expect to observe dynamical objects, such as clusters, resonances, jets – by some signal in the range of scales governed by the size of these objects. However, Andrzej and I were soon confronted with a question of interpretation: What means a continuous rise of the factorial moments with decreasing phase-space size? Indeed, no obvious scale emerged from the observed rise of factorial moments for the JACEE event (this feature was later on confirmed by systematic studies of various reactions).

At that time having no experience of such a phenomenon in high-energy physics, we were inspired by the studies of turbulent hydrodynamic flows[2, 3], in building a mathematical toy-model ($cf.$ α-model, in the terminology and the version of Ref.[3] which we used after modification[1]) reproducing a pattern of multiplicity fluctuations similar to what was surprisingly suggested by the JACEE event.

In its simplest version, for one event, the model is given by an (infinite) set of randomly independent numbers $\{W_s\}$ located along the branches of a Cayley tree structure (see Fig.1a), where s $(1 \le s \le \nu)$ is a branching step while k_s $(1 \le k_s \le K)$ denotes which of the K branches at a vertex bears the specific weight W_s. The multiplicity density profile is obtained for each individual bin by the following multiplication rule:

$$\rho_{[m]} = \prod_1^\nu W_s ,\qquad(2)$$

where the path $[m] = \{k_1, ..., k_s, ...k_\nu\}$ is uniquely associated to the phase-space bin m. Using the statistical independence of the weights and assuming

2

many steps $\nu \gg 1$, one gets:

$$\frac{\left\langle \rho_{[m]}^{q} \right\rangle}{\left\langle \rho_{[m]} \right\rangle^{q}} \;=\; \{W^q\}_r^{\nu} \;\equiv\; (M)^{\frac{\ln\{W^q\}_r}{\ln K}} \;, \tag{3}$$

where $M(=K^{\nu})$ is the total number of bins and, by definition:

$$\{W^q\}_r \;=\; \int r^{\cdot}(W)\, W^q\, dW \tag{4}$$

where $r(W)$ is a normalized weight probability distribution satisfying $\{1\}_r = \{W\}_r = 1$. As can be inferred from formulae (1-3), the multiplicity moments of an α-model (once the statistical noise is suitably de-convoluted) show up whith a power-law dependence on the number M of phase-space bins, e.g. the resolution with wich one examines the system. This behaviour is characteristic of the phenomenon of intermittency in hydrodynamical turbulence; it appears as a consequence of the random-cascading multiplicative property of the α-model.

Only a few years later, the ubiquous character of intermittent behaviour was recognised. In particular, an a-priori different type of cascading behaviour, –though much more familiar to particle physicists–, was suspected[4] and then shown[5] to possess intermittency properties. Interestingly enough, it is a direct consequence of Quantum Chromodynamics –the field theory of strong interactions – for the phase-space structure and development of quark and gluon jets. In this framework, the cascading mechanism can be called *additive*, since at each elementary vertex energy-momentum is conserved and fragmented among the decay products along the cascade. This is to be distinguished from the α-model, for which local densities are multiplied during the cascading process and thus not additively conserved . It is the subject of the present paper to exhibit the transformation which asserts the equivalence between *multiplicative* and *additive* cascading mechanisms and thus their identical intermittency properties.

3

2. From a local to a global description of intermittent cascading

As is explicited by the relation (3), the multiplicity density moments of the α-model unravel the structure of fluctuations in the local limit (i.e. short distance of order $1/M$). Our aim is now to look for the system as a whole, i.e., its global description. For this sake it is convenient[6] to introduce the (random) Partition function $P_f(q)$ and an associated generating function $\mathcal{Z}_\nu(u)$; One writes

$$P_f(q) \equiv \frac{1}{M} \sum_m \rho_{[m]}^q = \sum_{m=1}^M \prod_{s=1} \left(\frac{W_s}{K}\right)^q ; \ \mathcal{Z}_\nu(u) = \left\langle e^{-uP_f(1)} \right\rangle, \tag{5}$$

where one includes in the computation all the paths of the Cayley tree for ν cascading steps, see Fig.1a. Note that a thermodynamical formalism can be usefully introduced[6] where W_s/K acts like a Boltzmann weight and q as an inverse temperature.

Interestingly enough, $\mathcal{Z}_\nu(u)$ is known in Statistical Mechanics to obey a master equation[7]. Using an iterative procedure ($\nu \to \nu+1$) adding one new step at the beginning of the cascade and also the statistical independence of distinct sub-branching processes (and after some work), one obtains:

$$\mathcal{Z}_{\nu+1}(u) = \left\{ \mathcal{Z}_\nu^K \left(u\frac{W}{K} \right) \right\}_r. \tag{6}$$

The compact formula (6) will be at the root of the mathematical transformation between the additive and multiplicative versions of intermittent cascading. Let us for convenience introduce some interesting extensions of (6).

Extension # 1: Inserting $P_f(q)$ instead of $P_f(1)$ in the generating function (5), one gets a q-dependent master equation:

$$\mathcal{Z}_{\nu+1}(u,q) = \left\{ \mathcal{Z}_\nu^K \left(u \left(\frac{W}{K}\right)^q, q \right) \right\}_r \tag{6-1}$$

which exhibits a scaling behaviour in terms of q (or temperature[7].) Note that the equation derived from (6-1) for the first moment reproduces the intermittency property (3).

Extension # 2: The master equation can be easily extended to random-cascading processes including also random-branching (see Fig.1b). Introducing a time variable t, an ϵ probability of branching between t and $t + \epsilon$, a change of variables $\nu \to t$; $\nu + 1 \to t + \epsilon$, and going to the limit $\epsilon \to 0$ one gets

$$\frac{d\mathcal{Z}}{dt}(t; u) = \left\{ \mathcal{Z}^K \left(t; \frac{uW}{K} \right) \right\}_r - \mathcal{Z}(t; u) \tag{6-2}$$

Here too, the reader can check that the first moment equation obtained by functional derivation with respect to u leads to an intermittency property (3) (with a modified exponent). Note that Eq. (6-2) takes the familiar form of a gain-loss formula.

Extension # 3: Without modifying the basic properties of the cascading process, one may introduce a generalized distribution $r(t; W_1, W_2, ..., W_K)$. Using a suitable change of variables, and after some transformations of functions and variables one writes the following master equation:

$$\frac{d\mathcal{Z}}{dt}(t; u) = \left\{ \prod_{i=1}^{K} \mathcal{Z} \left(t, u \frac{W_i}{K} \right) \right\}_r - \mathcal{Z}(t; u). \tag{6-3}$$

We shall soon recognize that equation (6-3) provides a generic form of the additive cascading model of a jet based on QCD.

3. Additive vs. Multiplicative cascading models

In particle physics theory, however, intermittent behaviour has not been found directly under the form (6-1,3) of a multiplicative cascading process. It appears in the study of the multiplicity of gluons and quarks associated with an energetic jet in the framework of the resummed perturbative expansion of Quantum Chromodynamics. It has been for instance applied for the decays of $\mathcal{Z}°$s into quarks and gluons. In the leading-logs approximation of perturbative QCD for jet calculations, one writes[8] (for gluons):

$$\frac{\partial \mathcal{Z}}{\partial \ln Q}(Q, u) = \frac{1}{2} \int dz \ \Phi(Q, z) \ [\mathcal{Z}(Qz, u) \mathcal{Z}(Q(1 - z), u) - \mathcal{Z}(Q, u)] \tag{7}$$

5

where $\mathcal{Z}(Q, u)$ is the generating function of gluon multiplicity factorial moments from an initial gluon jet characterized by the virtuality Q; $\Phi(Q, z)$ is given in terms of the renormalized QCD coupling constant α_s and of the triple gluon Altarelli-Parisi-Kernel P_{GG}^G (including quarks would transform (7) into a two-by-two matrix form without changing our main conclusions). Note that the important QCD property of *angular ordering* allows the derivation of the multiplicity distribution for any subjet of initial energy E and conical aperture Θ starting from the same function $\mathcal{Z}(Q = E\Theta, u)$. As an important consequence, not only global but also local properties of multiplicity distributions of a QCD jet are determined by the solution of the equation (7). In particular angular ordering leads to the property of angular intermittency.[9]

As is explicit in formula (7), when compared to formulae(6-1,3), the gluon cascading process is generated by the fragmentation of energy-momentum between gluons at the vertex, using the energy-fraction variable z. Moreover, identifying the "time variable" with $\ln Q$, we find that Eq. (7) is *not* defined with equal-time observables (on contrary to Eqs. (6)), due to the mismatch between virtuality of a gluon and energy sharing.

Yet, the two approaches are equivalent, as can be inferred[10, 11] from a crucial property of the multiplicity distributions, namely KNO scaling [12]. This scaling property is verified in QCD[8] within the same conditions as Eq. (7). One may write, at least at high enough virtuality[1]; the following scaling relation:

$$\mathcal{Z}(Q, u) = \zeta\left(u \langle n \rangle_{\mathbf{Q}}\right), \tag{8}$$

where $\langle n \rangle_{\mathbf{Q}}$ is the average multiplicity at virtuality Q. Let us insert the KNO scaling relation (8) into (7). We get

$$
\begin{aligned}
\frac{\partial \mathcal{Z}}{\partial \ln Q} &= \frac{1}{2} \int dz \ \Phi(Q, z) \times \\
&\times \left[\zeta\left(u \langle n \rangle_{\mathbf{Qz}}\right) \zeta\left(u \langle n \rangle_{\mathbf{Q(1-z)}}\right) - \zeta\left(u \langle n \rangle_{\mathbf{Q}}\right) \right]
\end{aligned}
\tag{9}
$$

which, by a suitable change of variable can be cast into the following equivalent form: :

---

[1]There could appear some problems near the boundaries of the z-integration in Eq.(7). However a check of validity can be made using Monte-Carlo simulations [13].

$$\frac{\partial \mathcal{Z}(Q, u)}{\partial \ln Q} = \frac{1}{2} \int \int dW_1 \, dW_2 \, \tilde{r}(Q; W_1, W_2) \times$$

$$\times \left\{ \mathcal{Z}\left(Q, u\frac{W_1}{2}\right) \mathcal{Z}\left(Q, u\frac{W_2}{2}\right) - \mathcal{Z}(Q, u) \right\}, \tag{10}$$

by defining:

$$W_1 \equiv 2 \, \frac{\langle n \rangle_{\mathbf{Qz}}}{\langle n \rangle_{\mathbf{Q}}} \; ; \; W_2 \equiv 2 \, \frac{\langle n \rangle_{\mathbf{Q(1-z)}}}{\langle n \rangle_{\mathbf{Q}}} \tag{11}$$

and

$$\tilde{r}(Q; W_1, W_2) \equiv \Phi(Q; z) \frac{dz(Q; W_1)}{dW_1} \delta(W_2 - W_2(Q; W_1)), \tag{12}$$

where $z(Q; W_1)$ and $W_2(Q; W_1)$ are given in terms of the functional form of $\langle n \rangle_{\mathbf{Q}}$. This functional form is obtained by solving the linear equation coming from Eq.(7) for the first moment, i.e. the first derivative with respect to u. The solution of this equation is in general much simpler than for the generating function or higher moments. Note that a solution of this simpler equation is sufficient to define the transformation we look for (and thus to derive the intermittency properties).

Equations (10-12) are to be compared with the generic equation (6-3). More precisely, the change of function and variable

$$\left(\int \int dW_1 \, dW_2 \, \tilde{r}(Q; W_1, W_2) \right)^{-1} \tilde{r}(Q; W_1, W_2) \quad \rightarrow \quad r(Q; W_1, W_2)$$

$$\left(\int \int dW_1 \, dW_2 \, \tilde{r}(Q; W_1, W_2) \right) \partial \ln Q \quad \rightarrow \quad \partial t, \tag{13}$$

gives the equivalence of gluon cascading with a random branching, random cascading, *multiplicative* process. It amounts to choosing in Eq. (6-3) $K = 2$ and the specific probability distribution r, see (13). In a sense, the *multiplicative* formulation provides a statistical description of the multiplicity density distribution at equal time (or virtuality, or jet aperture), while the *additive* one describes the sharing of energy-momentum. Both descriptions are intimately connected by the KNO scaling property. It is worth noticing that this

specific stochastic process is, strictly speaking, of *semi-random* type since the value taken by the random weight W_1 fixes the other one W_2 at the same vertex.

4. Application to QCD gluon cascading

Let us consider QCD cascading at the leading logarithmic approximation, by keeping into account the effect of energy-momentum conservation at the vertex. This problem has been recently raised for the fluctuation pattern and solved[11]. In terms of *global* observables, it amounts to solving equation (7) with the kernel $\Phi(Q^2, z) \equiv 2\gamma_0^2/z$ where $\gamma_0^2 \equiv 2\alpha_s N_c/\pi$ is kept fixed for sake of simplicity. Using the formulation of Eqs. (10-13), one gets:

$$\frac{\partial \mathcal{Z}(Q, u)}{\partial \ln Q} = \gamma_0 \int \frac{dW_1}{W_1} \times$$

$$\times \left[\mathcal{Z}\left(Q, \frac{uW_1}{2}\right) \mathcal{Z}\left(Q, u\left[1 - \left(\frac{W_1}{2}\right)^{\gamma_0^{-1}}\right]^{\gamma_0}\right) - \mathcal{Z}(Q, u) \right], \tag{14}$$

where the relations:

$$W_1 \equiv 2z^{\gamma_0} \; ; \; W_2 = 2 \; (1 - z)^{\gamma_0}, \tag{15}$$

are coming from the expression of the mean multiplicity $\langle n \rangle_{\mathbf{Q}} \propto Q^{\gamma_0}$; by inversion, one finds:

$$z \equiv z(W_1) = \left(\frac{W_1}{2}\right)^{1/\gamma_0} \; ; \; W_2 = 2 \left[1 - \left(\frac{W_1}{2}\right)^{1/\gamma_0}\right]^{\gamma_0},$$

$$\tilde{r}(Q; W_1, W_2) \equiv \gamma_0 \frac{1}{W_1} \delta\left(W_2 - 2\left[1 - \left(\frac{W_1}{2}\right)^{1/\gamma_0}\right]^{\gamma_0}\right). \tag{16}$$

A few remarks are in order about the master equation (14).

i) Comparing (14) with Eqn.(6-3) shows that $\mathcal{Z}$ is the generating function for a random-branching process with triple vertex $1 \to 2$, (up to transformations like (13)). At each vertex, one branch corresponds to a randomly

8

chosen weight W_1 with the probability law γ_0/W_1, see expressions (16). The second-branch weight is then determined by $W_2 = 2\left[1 - (W_1/2)^{1/\gamma_0}\right]^{\gamma_0}$. Interestingly enough, the local constraint of energy-momentum conservation along the additive cascading process is transposed into a specific stochastic law at the multiplicative vertex of the density cascading process.

ii) In the approximation $W_1 < 2$; $\gamma_0 \ll 1$, then $\left(\frac{W_1}{2}\right)^{1/\gamma_0} \ll 1$, and one can write a simplified version of equation (13) namely:

$$\frac{\partial \ln \mathcal{Z}(Q,u)}{\partial \ln Q} = \gamma_0 \int \frac{dW_1}{W_1} \left(\mathcal{Z}\left(Q, \frac{uW_1}{2}\right) - 1\right). \qquad (17)$$

Equation (17) has been found[9] equivalent to the double-leading-log approximation of QCD for the jet process, where one can neglect the recoil effect upon the leading parton (quark or gluon). However, sizeable values of γ_0 (which is still of order .5 at LEP energies) do lead to substantial modifications of the multiplicity distributions[14] and their fluctuations in phase-space[11] due to energy-momentum conservation.
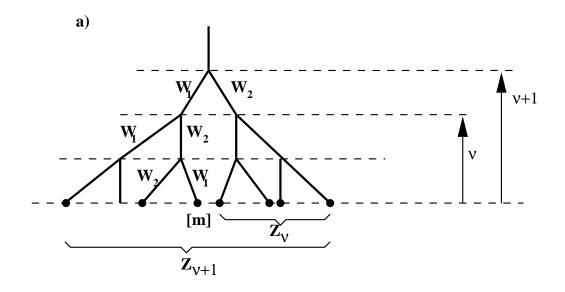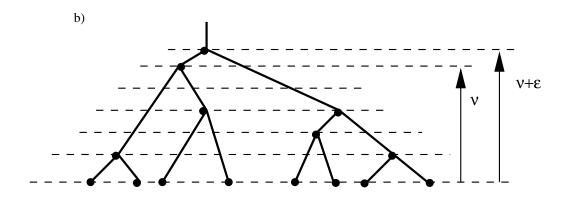
Figure Caption

a) Fixed branching
Each path $[m]$ corresponds to a bin in phase space. The fluctuation density in the bin is described by the product of random weights W along the path $[m]$. In the case of "semi-randomness" the decay is assymetric at each step: one weight W_1 follows a random law, while W_2 is a function of W_1 for the same vertex (W_1 or W_2 can be randomly left or right). The tree can be separated into 2 branches. Each of these branches defines in average $\mathcal{Z}_\nu$ of the Partition function, the whole tree itself corresponding to $\mathcal{Z}_{\nu+1}$.

b) Random branching

Same structure as in Figure 1.a, but with small increments ε of the number of steps ν and random-branching with uniform probability ε.

a)

W_1 W_2

W_1 W_2

W_2 W_1

[m]

Z_ν

$Z_{\nu+1}$

$\nu+1$

ν

b)

$\nu+\varepsilon$

ν

Figure

References

[1] A. Bialas and R. Peschanski, *Nucl. Phys.* **B273** (1986) 703, **B308** (1988) 857, *Phys. Lett.* **B207** (1988) 59.

[2] B.B. Mandelbrot, *J. Fluid. Mech.* **62** (1974) 331; U. Frisch, P.L. Sulem and M. Nelkin, *J. Fluid Mech.* **87** (1978) 718;

[3] D. Schertzer and S. Lovejoy, in "Turbulent Shear flows 4" eds. L. J.S. Bradbury et al. (Springer, 1984).

[4] L. Van Hove, private communication (1989);

[5] Y. Dokshitzer and I. Dremin, *Nucl. Phys.* **B402** (1993) 139; R. Peschanski, in "Proceedings of the XXII International Symposium on Multiparticle Dynamics" Santiago de Compostela, Spain, July 1992 (C. Pajares editor, World Scientific) P. 164; P.Brax, J.-L. Meunier and R. Peschanski, *Zeit. für Physik* **C62** (1994) 649; W. Ochs and I. Wosiek, *Phys. Lett.* **B305** (1993) 144.

[6] R. Peschanski, *Nucl. Phys.* **B327** (1989) 144; in a different but equivalent language, see: A. Bialas, A. Szczerba and K. Zalewski, *Zeit. für. Phys.* **C46** (1990) 163.

[7] B. Derrida and H. Spohn, *J. Stat. Phys.* **51** (1988) 817, and references therein.

[8] See, for instance, the book: "Basics of Perturbative QCD" Yu. L. Dokshitzer, V.A. Khoze, A.H. Mueller and S.I. Troyan (Editions FRONTIERES, FRANCE), p. 124 - 169, and references therein.

[9] see P. Brax et al. in [5].

[10] I thank J.-L. Meunier for having first pointed out to me the relation between the two formalisms of intermittent behaviour by the generating function.

[11] J. -L. Meunier and R. Peschanski, "Energy Conservation Constraints on Multiplicity Correlations in QCD jets", hep-ph/9603294 to appear in *Zeit. für Phys.* **C**.

[12] Z. Koba, H.B. Nielsen and P. Olesen, *Nucl. Phys.* **B40** (1972) 317; On the rôle of KNO scaling in the α-model see: Y. Gabellini, J.-L. Meunier and R. Peschanski, *Zeit. für Phys.* **C55** (1992) 455.

[13] J.-L. Meunier, private communication.

[14] See, for instance, Yu.L. Dokshitzer, *Phys. Lett.* **B305** (1993) 295. See also F. Cuypers, *Zeit. fur Phys.* **C54** (1992) 87.